Analytic Solutions for Pricing Double Barrier Options in the Presence of Stochastic Volatility

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Task

- There exist closed-form solutions for vanilla options in the presence of stochastic volatility for nearly a decade (Heston, 1993)

- For double barrier options we still depend on slow numerical methods, in particular Finite Difference or Monte Carlo methods

- Lipton (2001) obtains semi-analytical solutions for a special case

Overview

1. The Heston Model
2. Free Green's Function
3. Method of Images
4. Eigenfunction Expansion
5. Application to other options
6. Implementation

1. The Heston Model

1.1 Overview

- Variance of the spot process is no longer deterministic, but modelled as a second stochastic process

- Heston chose a mean-reverting square-root process, i.e.

$$dS_t = S_t\Big(\mu dt + \sqrt{v(t)}dW_t^{(1)}\Big)$$
$$dv_t = \kappa\Big(\theta - v(t)\Big)dt + \sigma\sqrt{v(t)}dW_t^{(2)}$$

  where $W_t^{(1)}$, $W_t^{(2)}$ are Brownian Motions with correlation ρ.

1.2 Differential Equation

Every contingent claim $U(t, v, S)$ paying $g(S) = U(T, v, S)$ has to satisfy the partial differential equation

$$U_t + (r_d - r_f)SU_S + \frac{1}{2}\sigma^2 vU_{vv} + \frac{1}{2}vS^2U_{SS} +$$
$$\rho\sigma vSU_{vS} - r_dU + \left(\tilde{\kappa}(\tilde{\theta} - v) - \lambda v\right)U_v = 0$$

(Can be shown by creating a replication portfolio, if we assume the market price of volatility risk λ has the form $\lambda(t, v, S) = \lambda v$).

After reverting the time, i.e. applying the transformation

$$\tau = T - t$$

and elimination of the market price of volatility risk λ by replacing $\tilde{\kappa}$ and $\tilde{\theta}$ by $\kappa := \tilde{\kappa} + \lambda$ and $\theta := \frac{\tilde{\kappa}\tilde{\theta}}{\tilde{\kappa}+\lambda}$ one can rewrite it as

$$\boxed{\begin{aligned} U_\tau - (r_d - r_f)SU_S - \frac{1}{2}\sigma^2 vU_{vv} - \frac{1}{2}vS^2U_{SS} - \\ \rho\sigma vSU_{vS} + r_dU - \kappa(\theta - v)U_v = 0 \end{aligned}}$$

1.3 Double Barrier Options

We concentrate on double-barrier knock-out calls, which have to satisfy the pde with conditions

$$\begin{aligned} U(T, v, S) &= (S - K)^+ \\ U(t, v, A) = U(t, v, B) &= 0 \end{aligned}$$

respectively

$$\boxed{\begin{aligned} U(0, v, S) &= (S - K)^+ \\ U(\tau, v, A) = U(\tau, v, B) &= 0 \end{aligned}}$$

2. Free Green’s Function

2.1 Overview

- Transform the pricing problem into a more tractable form
- Then derive the transition probability density function for the problem without barriers.

2.2 Transformation

We want to write the derived pricing problem in forward terms to remove the interest rates.

Apply the transformation

$$\widehat{U}(\tau, v, F) = \widehat{U}(\tau, v, e^{(r_d - r_f)\tau} S) = U(\tau, v, S) e^{r_d \tau}$$

Results in

$$
\begin{aligned}
\widehat{U}_\tau - \frac{1}{2} v F^2 \widehat{U}_{FF} - \rho\sigma v F \widehat{U}_{vF} - \frac{1}{2}\sigma^2 v \widehat{U}_{vv} - \kappa(\theta - v)\widehat{U}_v = 0 \\
\widehat{U}(0, v, F) = (F - K)^+ \\
\widehat{U}(\tau, v, \widehat{A}) = \widehat{U}(\tau, v, \widehat{B}) = 0 \\
\widehat{U}_\tau(\tau, 0, F) - \kappa\theta \widehat{U}_v(\tau, 0, F) = 0
\end{aligned}
$$

with transformed barriers

$$
\widehat{A} = e^{(r_d - r_f)\tau} A, \quad \widehat{B} = e^{(r_d - r_f)\tau} B
$$

We now nondimensionalize the problem by representing $\widehat{U}$ as

$$\widehat{U}(\tau, v, F) = K\Phi(\tau, v, \xi)$$

with $\xi = F/K$ and then further reduce the problem via the transformation

$$\begin{aligned} \tau \to \tau, \quad \xi \to X = \ln(\xi) \\ \Phi(\tau, v, \xi) \to W(\tau, v, X) = e^{-X/2}\Phi(\tau, v, \xi) \end{aligned}$$

This results in the following pricing problem:

$$W_\tau - \frac{1}{2}vW_{XX} - \frac{1}{2}\sigma^2 vW_{vv} - \rho\sigma vW_{vX} - \widehat{\kappa}(\widehat{\theta} - v)W_v + \frac{1}{8}vW = 0$$

$$W(0, v, X) = (e^{X/2} - e^{-X/2})^+$$

$$W(\tau, v, \widehat{a}) = W(\tau, v, \widehat{b}) = 0$$

where

$$\widehat{a} = \ln(\widehat{A}/K), \quad \widehat{b} = \ln(\widehat{B}/K)$$

$$\widehat{\kappa} = \kappa - \sigma\rho/2, \quad \widehat{\theta} = \kappa\theta/\widehat{\kappa}$$

One can show that this is the backward Kolmogoroff equation * with killing at a rate of $\frac{v}{8}$ for the following system of stochastic differential equations:

$$\begin{aligned} dX &= \sqrt{v}dW_t^{(1)} \\ dv &= \widehat{\kappa}(\widehat{\theta} - v)dt + \sigma\sqrt{v}dW_t^{(2)} \end{aligned}$$

*see Karatzas and Shreve (1991)

2.3 Derivation

Once we found the Green's function of our pricing problem, we can use the following theorem to obtain a solution:

Theorem 1. *Let* $p(t, X, v, t', X', v')$ *be the Green's function of a process* (X_t, v_t) *and let* $w(X, v)$ *be the payoff function. The pricing formula* $W(t, X, v)$ *can then be expressed as*

$$W(t, v, X) = \int\int p(t, X, v, T, X', v')w(X', v')dX'dv'$$

In our case we can make use of the fact, that the coefficients and the killing rate are independent of t, X and can hence concentrate on the differences $\tau = t' - t$ and $Y = X' - X$ rather than on the arguments themselves:

$$p(t, X, v, t', X', v') = p(\tau, Y, v, v')$$

Another simplification can be done, when we observe that the payoff of our double barrier call is independent of the volatility. Hence we only need to find its integral over v', i.e.

$$q(\tau, Y, v) = \int_0^\infty p(\tau, Y, v, v') dv'$$

Due to the fact that the coefficients of q are linear functions of v, we can now guess the following form:

$$q(\tau, Y, v) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{ikY + 2(\mathbf{A}(\tau,k) - \mathbf{B}(\tau,k)v)/\sigma^2} dk$$

After some calculations we obtain:

$$\mathbf{A}(\tau,k) = -\kappa\theta(\mu+\zeta)\tau - \kappa\theta \ln\left(\frac{-\mu+\zeta+(\mu+\zeta)e^{-2\zeta\tau}}{2\zeta}\right)$$

$$\mathbf{B}(\tau,k) = \frac{\sigma^2(k^2+1/4)(1-e^{-2\zeta\tau})}{4\Big(-\mu+\zeta+(\mu+\zeta)e^{-2\zeta\tau}\Big)}$$

where

$$\mu(k) = -\frac{1}{2}(ik\sigma\rho+\widehat{\kappa})$$

$$\zeta(k) = \frac{1}{2}\sqrt{k^2\sigma^2(1-\rho^2)+2ik\sigma\rho\widehat{\kappa}+\widehat{\kappa}^2+\frac{\sigma^2}{4}}$$

3. Method of Images

3.1 Overview

Now: Incorporate the barriers into the Free Green's function via the method of images.

Principle: Derive a generalized reflection principle and use it to express the restricted problem as an infinite sum of the unrestricted one.

References: Dewynne, Howison and Wilmott (1993); Andreasen (2001).

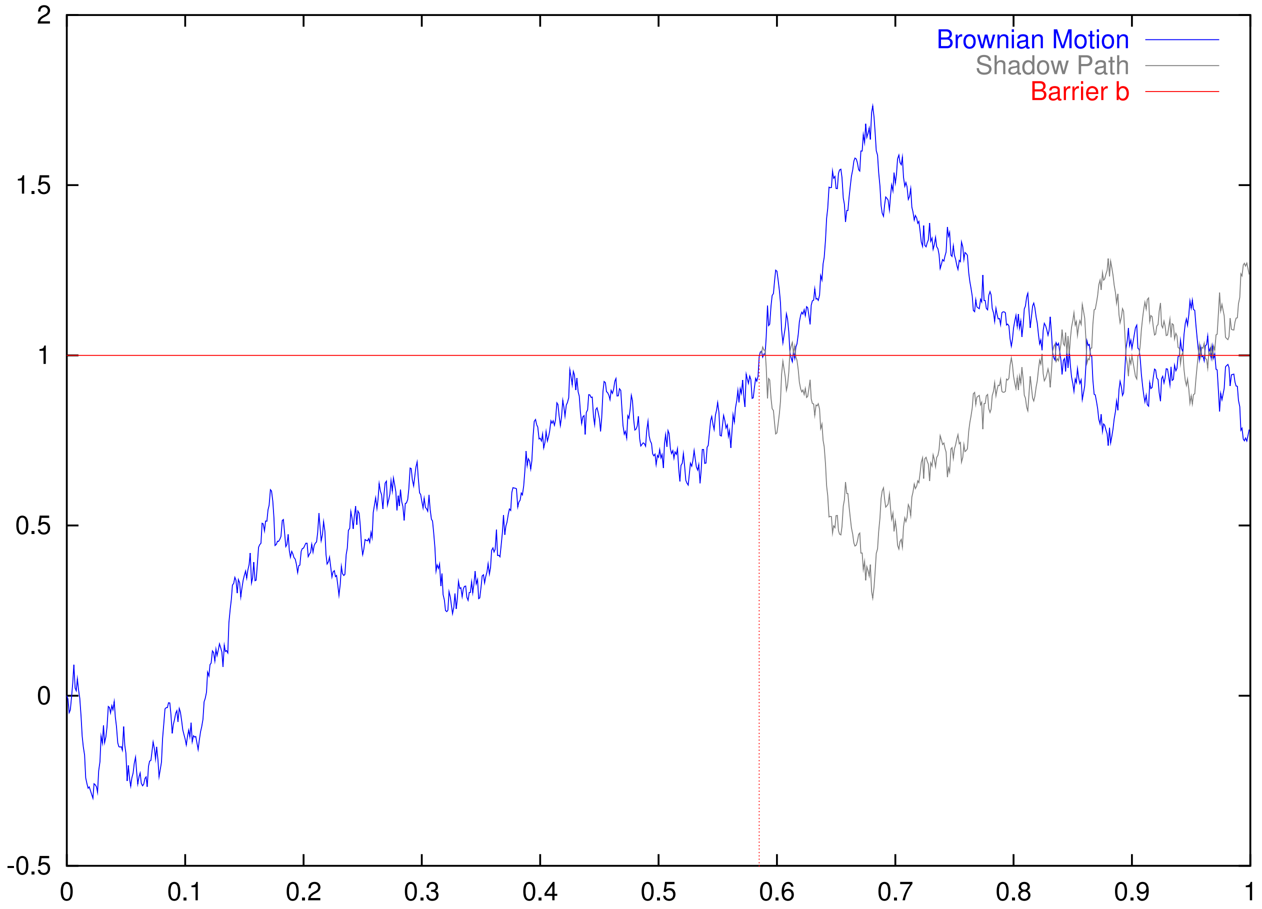
3.2 Reflection Principle

Theorem 2 (Reflection Principle). *Let* $b > 0$ *and* W_t *a Brownian motion. Let* T_b *be the first passage time with respect to level* b *defined as*

$$T_b := \inf_{t \geq 0} \Big(W_t = b \Big)$$

Then we can express the joint distribution of first passage time and Brownian motion directly with the distribution of the Brownian motion. In particular we have

$$P\Big(T_b < t, W(T) < x\Big) = P\Big(W(T) > 2b - x\Big) \quad \forall x < b$$


Brownian Motion
Shadow Path
Barrier b
2
1.5
1
0.5
0
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

A generalization of the Reflection Principle to a situation with two barriers leads to

Lemma 3 (Bounded Green's Function). *Let* $\rho = 0$*. Then*

$$\begin{aligned} & G(\tau, X, v, X') \\ & \quad = \sum_{-\infty}^{\infty} \Big(q(\tau, X' - X_n, v) - q(\tau, X' + X_n - 2a, v) \Big) \end{aligned}$$

where $X_n = X + 2n(b - a)$*.*

3.3 Limitations

- One should note, that the theorem relies heavily on the symmetry of the Brownian Motion.

- As the lemma is a direct consequence of the theorem, we have to check if the free Green's function in our problem is indeed even in Y.

- This is only true for the case $\rho = 0$: Then $q(\tau, Y, v)$ is invariant with respect to the reflections $Y, k \to -Y, -k$. Else the distribution is skewed and we cannot apply the lemma any longer.

3.4 Uncorrelated Case

In the special case with $\rho = 0$ we can nevertheless use this method in a straightforward manner. First of all we know that $N \equiv 0$.

The Green's function keeps it basic form, only the functions μ and ζ reduce to

$$\begin{aligned} \mu(k) &= -\frac{1}{2}\kappa \\ \zeta(k) &= \frac{1}{2}\sqrt{k^2\sigma^2 + \kappa^2 + \frac{\sigma^2}{4}} \end{aligned}$$

As seen before we can now write the solution of the pde as

$$W(\tau, v, X) = \int_{\widehat{a}}^{\widehat{b}} G(\tau, X, v, X')W(0, v, X')dX'$$

and after using the suitable payoff function as

$$W(\tau, v, X) = \int_{\widehat{a}}^{\widehat{b}} G(\tau, X, v, X')(e^{X'/2} - e^{-X'/2})^{+}dX'$$

Performing the necessary substitutions yields the following result:

$$W(\tau, v, X) = \sum_{n=-\infty}^{\infty} W_n(\tau, v, X)$$

where

$$\begin{aligned} W_n(\tau, v, X) \;\; = \;\; & e^{\widehat{b}/2}\mathbf{H}^{+}(\tau, X_n - \widehat{b}, v) - \mathbf{H}^{+}(\tau, X_n, v) \\ & - e^{-\widehat{b}/2}\mathbf{H}^{-}(\tau, X_n - \widehat{b}, v) + \mathbf{H}^{-}(\tau, X_n, v) \\ & - e^{-\widehat{b}/2}\mathbf{H}^{+}(\tau, X_n - 2\widehat{a} + \widehat{b}, v) + \mathbf{H}^{+}(\tau, X_n - 2\widehat{a}, v) \\ & + e^{\widehat{b}/2}\mathbf{H}^{-}(\tau, X_n - 2\widehat{a} + \widehat{b}, v) - \mathbf{H}^{-}(\tau, X_n - 2\widehat{a}, v) \end{aligned}$$

and

$$\mathbf{H}^{\pm}(\tau, X, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ikX + 2(\mathbf{A}(\tau,k) - \mathbf{B}(\tau,k)v)/\sigma^2}}{ik \pm 1/2}$$

3.5 Summary

- The method works only in the special case and cannot be directly generalized.

- The method yields formulas, which are computationally expensive and somehow inelegant.

4. Eigenfunction Expansion

4.1 Overview

Second approach: eigenfunction expansion.

Principle: Use separation of variables to write the solution of the partial differential equation as infinite series of its eigenfunctions. Replacing the final payoff with a Dirac δ-function results then in the bounded Green's function.

References: Karlin and McGregor (1960); Lewis (2000)

4.2 Constant Volatility

Let us consider for the moment the classical Black-Scholes setting. At first we have to transform the basic pricing problem

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r_f - r_d) S V_S - r_d V = 0$$

in its nondimensional form, first by substituting

$$\chi = \sigma^2 \tau$$

to get the form

$$V_\chi - \frac{1}{2} S^2 V_{SS} - \frac{(r_f - r_d)}{\sigma^2} S V_S + \frac{r_d}{\sigma^2} V = 0$$

Then perform the transformation

$$V(\chi, S, K) \to K\Theta(\chi, \zeta), \quad \zeta = S/K$$

to get the form

$$\Theta_\chi - \frac{1}{2}\zeta^2\Theta_{\zeta\zeta} - \frac{(r_f - r_d)}{\sigma^2}\zeta\Theta_\zeta + \frac{r_d}{\sigma^2}\Theta = 0$$

followed by introducing $X = \ln(\zeta)$ to obtain

$$\Theta_\chi - \frac{1}{2}\Theta_{XX} + \Big(\frac{1}{2} - \frac{1}{\sigma^2}(r_f - r_d)\Big)\Theta_X + \frac{r_d}{\sigma^2}\Theta = 0$$

- Finally make a change of the dependent variable

  $$U(\chi, X) = e^{(\frac{r}{\sigma^2}+\gamma_-^2/2)\chi+\gamma_- X}\Theta(\chi, X)$$

  where

  $$\gamma_\pm = \pm\frac{1}{2} + \frac{r_f - r_d}{\sigma^2}$$

- The transformed problem is then just the heat equation

  $$U_\chi - \frac{1}{2}U_{XX} = 0$$

  with boundary conditions

  $$U(\chi, a) = 0 = U(\chi, b)$$

- We now separate of variables via

$$U(\chi, X) = e^{-\mu\chi} g(X)$$

- This results in

$$2\mu g + g_{XX} = 0$$

with adjusted boundary conditions

$$g(a) = 0$$
$$g(b) = 0$$

- After replacing 2μ by k^2 we can write the well-known general solution to this simple harmonic motion equation with boundary conditions as

$$g(X) = \Phi \cos(kX) + \Psi \sin(kX)$$

- After shifting the function via $X \to X - a$ we can use the boundary conditions to determine the coefficients: $g(a) = 0$ implies $\Phi = 0$ and $g(b) = 0$ gives us a condition on k to get a nontrivial solution:

$$k = k_n = \frac{\pi n}{b - a}$$

- Now we can write the eigenvalues μ_n simply as

$$\mu_n = \frac{1}{2}k_n^2 = \frac{\pi^2 n^2}{2(b-a)^2}$$

- Using this result we get the following form of the eigenfunctions:

$$g_n(X) = \Psi_n \sin\Big(k_n(X-a)\Big)$$

- Combining our results we get the solution of the time-dependent problem

$$U_n(\chi, X) = e^{-\mu\chi} g_n(X) = e^{-\frac{1}{2}k_n^2\chi} \Psi_n \sin\Big(k_n(X-a)\Big)$$

- Hence the general solution as linear combination of the $U_n(\chi, X)$ has the form

$$U(\chi, X) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}k_n^2\chi} \Psi_n \sin\Big(k_n(X-a)\Big)$$

Theorem 4 (Orthogonality of the Eigenfunctions). *Let* $\aleph =$ $\mathsf{L}^2([a,b])$ *be the Hilbert space of square-integrable functions on* $[a,b]$*, where the inner product is defined as*

$$\langle f, g\rangle = \int\limits_a^b f(X)g(X)dX$$

Then the eigenfunctions g_n *are orthogonal and we have*

$$\int\limits_a^b g_n(X)g_m(X)dX = \delta_{nm}\Big(\frac{1}{2}\Psi_n^2(b-a)\Big)$$

- For the remaining task of determining the coefficients we can make use of the fact that

$$u(X) = U(0, X) = \sum_{n=1}^{\infty} \Psi_n \sin\Big(k_n(X-a)\Big)$$

- In combination with the orthogonality conditions we get

$$\int_a^b g_n(X) u(X) dX \quad = \Psi_n^2 \tfrac{b-a}{2}$$

- Therefore the coefficients Ψ_n can be written as

$$\Psi_n = \frac{2 \int_a^b u(X) \sin\Big(k_n(X-a)\Big) dX}{b-a}$$

- Get the Green's Function by choosing a Dirac δ-function as payoff function $u(X)$:

$$\begin{aligned}\Psi_n &= \frac{2\int\limits_a^b \delta(X-X')\,\mathsf{sin}\Big(k_n(X-a)\Big)dX}{b-a}\\ &= \frac{2}{b-a}\,\mathsf{sin}\Big(k_n(X'-a)\Big)\end{aligned}$$

- Hence the Green's function has the form

$$G(\chi,X,X') = \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\frac{1}{2}k_n^2\chi}\,\mathsf{sin}\Big(k_n(X-a)\Big)\,\mathsf{sin}\Big(k_n(X'-a)\Big)$$

The solution of the pricing problem can now again be written as convolution of the Green's function with the payoff function

$$U(\chi, X) = \int_a^b G(\chi, X, X')u(X')dX'$$

4.3 Stochastic Volatility

- Since the same transformations are not applicable in the stochastic volatility case, we would need to find a suitable transformation if we want to proceed in analogy.

- To avoid unnecessary calculations we will therefore look directly at the free Green's function obtained in the previous chapter. We can rewrite it as

$$G(\tau, X, v, X') = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \Big(\Psi \sin \big(k(X - X') \big) + \Theta \cos \big(k(X - X') \big) \Big) \times$$

$$e^{2(\mathbf{A}(\tau,k) - \mathbf{B}(\tau,k)v)/\sigma^2} dk$$

Now we can use the same arguments as previously:

- The boundary condition $G(\tau, X, v, \widehat{a}) = 0$ yields $\Theta = 0$.
- The second boundary condition $G(\tau, X, v, \widehat{b}) = 0$ yields
$$k = k_n = \frac{\pi n}{\widehat{b} - \widehat{a}}$$
- Using the orthogonality properties to obtain the coefficients Ψ_n results in $\Psi_n = \frac{2}{\widehat{b}-\widehat{a}} \int\limits_{\widehat{a}}^{\widehat{b}} u(X) \sin\Big(k_n(X - \widehat{a})\Big) dX$

We then obtain the bounded Green's function by setting $u(X) = \delta(X - X')$:

$$G(\tau, X, v, X') = \frac{2}{\widehat{b} - \widehat{a}} \sum_{n=1}^{\infty} e^{2(\mathbf{A}(\tau,k) - \mathbf{B}(\tau,k)v)/\sigma^2} \times \\ \sin\left(k_n(X - \widehat{a})\right) \sin\left(k_n(X' - \widehat{a})\right)$$

- As before we get our transformed pricing formula via

$$W^{(DBC)}(\tau, v, X) = \int_{\widehat{a}}^{\widehat{b}} G(\tau, X, v, X')(e^{X'/2} - e^{-X'/2})^{+} dX'$$

- This can be written as

$$W^{(DBC)}(\tau, v, X) = \sum_{n=1}^{\infty} e^{2(\mathbf{A}(\tau,k) - \mathbf{B}(\tau,k)v)\sigma^2} \varphi_n^{(DBC)} \sin\left(k_n \ln(F/\widehat{A})\right) dX'$$

where

$$\varphi_n^{(DBC)} = \frac{2\left((-1)^{n+1} k_n (\sqrt{\widehat{B}/K} - \sqrt{K/\widehat{B}}) + \sin\left(k_n \ln(\widehat{A}/K)\right)\right)}{(k_n^2 + 1/4) \ln(\widehat{B}/\widehat{A})}$$

We are now only left with the task of undoing the previous transformations. Let us therefore apply the back-transformation

$$U^{(DBC)}(\tau, S, v) = e^{-r_d \tau + X/2} K W(\tau, X, v), \quad X = \ln(F/K)$$

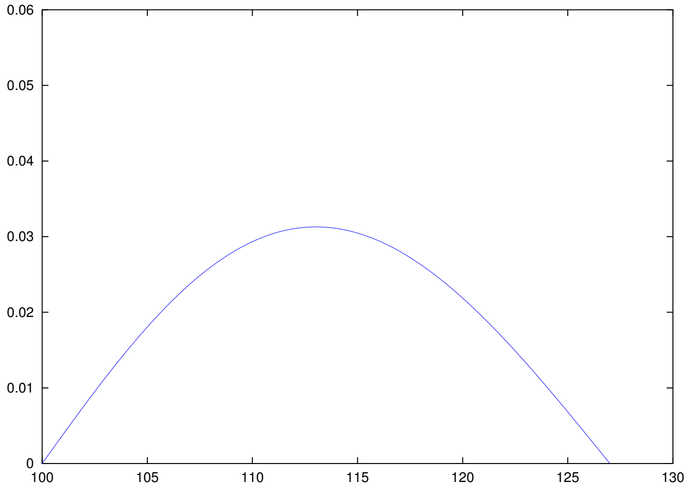
and obtain as final pricing formula

$$U^{(DBC)}(\tau, S, v) = e^{-r_d \tau} \sqrt{FK} \sum_{n=1}^{\infty} e^{2(\mathbf{A}(\tau, k_n) - \mathbf{B}(\tau, k_n) v)/\sigma^2} \times \varphi_n^{(DBC)} \sin(k_n \ln(F/\widehat{A}))$$

4.4 Limitations

Now we can see, why a generalization of Lipton's special case cannot lead to equations of the same form:

- If we introduce a correlation $\rho \neq 0$ we would obtain a W_{vX} term in the partial differential equation. (cosine-terms!)
- Even a choice of $r_d \neq r_f$ would cause problems, since this results in time-dependent barriers. (Separation of Variables makes no sense!)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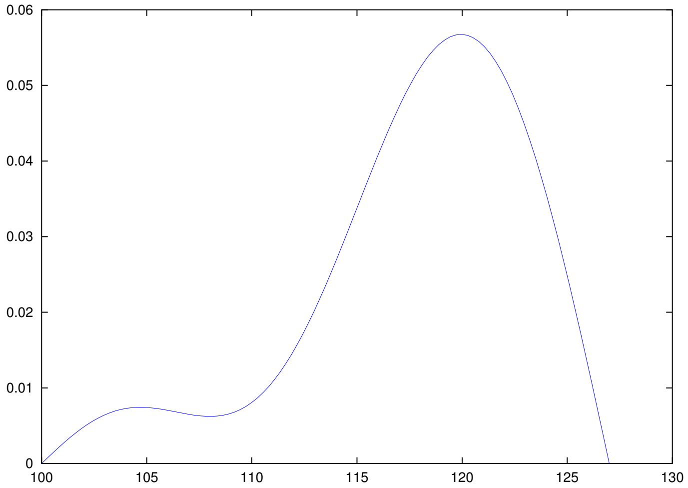
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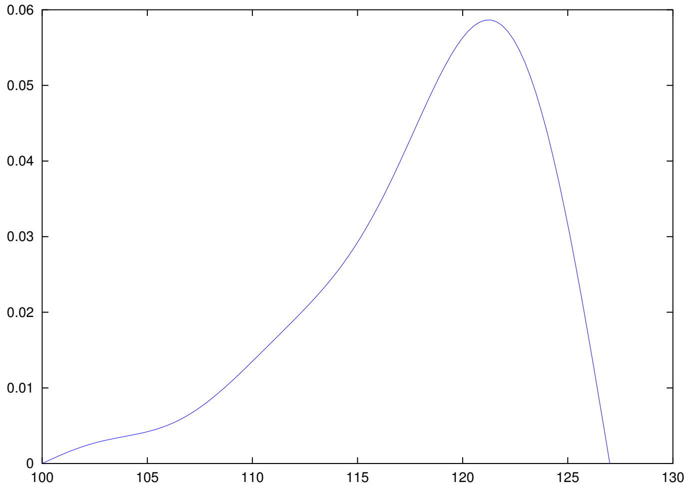
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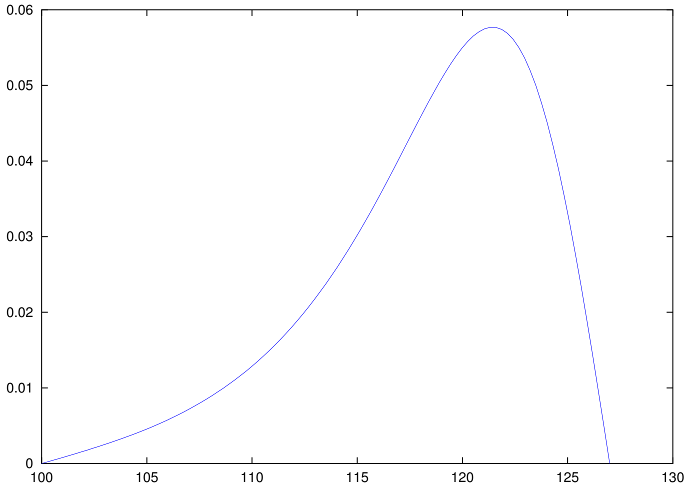
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4.5 Summary

- The method works only in the special case and cannot be directly generalized.

- The method yields elegant formulas, which can be easily implemented.

5. Application

5.1 Overview

We can use several methods to adopt this approach to options with a different payoff scheme:

- Solving the Integral: DNT, Digitals
- Replication Portfolios: DOT, Knock-Ins
- Asymptotic Approximation: Vanillas, Single Barriers, Power Options

6. Implementation

6.1 Overview

- The Eigenfunction Expansion Method was implemented in Mathematica 3.0
- As benchmarks we used the analytic solutions for vanillas and the finite-difference method for barriers.
- Lower and upper artificial barriers at 1 and 500 were introduced, if the need arose.

6.2 Parameters

Parameter	Value
κ	1.98937
r_d	0.036814
r_f	0.036814
σ	0.33147
S_0	123.4
K	120
v_0	0.014328
τ	0.50137
θ	0.011876
ρ	0

6.3 Vanillas

Type	Vanilla	Vanilla	Digital	Power
P/C	Call	Put		Call
V_{Ana}	5.56965	2.23072		
V_{Eigen}	5.56755	2.22973	0.639612	33.2701
Deviation (in %)	0.04	0.04		
Number of Iterations	302	342	164	302
Elapsed Time	0.40	0.47	0.22	1.172

6.4 Barriers

Type	Up and Out	Double Barrier	DNT
P/C	Call	Call	
Lower Barrier		120	120
Upper Barrier	127	127	127
V_{FD}	0.327446	n.a.	n.a.
V_{Eigen}	0.325321	0.109482	0.0317398
Number of Iterations	168	4	4
Elapsed Time	0.23	0.02	< 0.001

6.5 Efficiency

The method performs well, if

- the barriers are narrow
- the maturity is long (!)